Stochastic model related to the Klein-Gordon equation

Alexander V. Plyukhin and Jeremy Schofield
Chemical Physics Theory Group, Department of Chemistry, University of Toronto, Toronto, Ontario, Canada M5S 3H6
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In this note, a one-dimensional system composed of an assembly of interacting particles, is considered. Each particle in the assembly moves in a well-defined trajectory on a line with a velocity of fixed magnitude, which randomly reverses direction. The intrinsic forces in the system are assumed to induce stochastic transitions between velocity states in such a way that the average dynamics of the assembly is Newtonian. It is shown that there is a close analogy between collective oscillations in the model system and the propagation of a free quantum particle in one dimension.

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In his book on path integrals, Feynman [1] considered a one-dimensional, two-state stochastic process in which a particle is constrained to move on a line with a velocity of fixed value c whose direction of motion occasionally changes by 180°. Because of its zig-zagging path in two-dimensional space time, the process is sometimes referred to as the checkerboard model. Feynman found that the properly weighted summation of all such paths leads to the exact propagator for the one-dimensional Dirac equation. The underlying differential equations of the checkerboard process have been considered by Gaveau *et al.* [2]. Assuming that the reversals of direction are random and Poisson distributed, the probability densities $\phi_+(x,t)$ and $\phi_-(x,t)$ for a particle at position x at time t and moving to the right and to the left, respectively, satisfy

$$\frac{\partial\phi_+}{\partial t}=-c\frac{\partial\phi_+}{\partial x}-w(\phi_+-\phi_-), \tag{1}$$

$$\frac{\partial\phi_-}{\partial t}=c\frac{\partial\phi_-}{\partial x}+w(\phi_+-\phi_-), \tag{2}$$

where w is the rate of transitions between two velocity states. Equations (1) and (2) are the continuous version of a persistent random walk [3] leading to the telegrapher's equation

$$\frac{\partial^2\phi_\pm}{\partial t^2}-c^2\frac{\partial^2\phi_\pm}{\partial x^2}=-2w\frac{\partial\phi_\pm}{\partial t}. \tag{3}$$

In matrix form, one can express Eqs. (1) and (2) as

$$\left(\frac{\partial}{\partial t}+w\right)\Phi=-c\sigma_3\frac{\partial\Phi}{\partial x}+w\sigma_1\Phi, \tag{4}$$

where Φ is the two-component column vector $(\phi_+,\phi_-)^T$ and σ_1,σ_3 are the Pauli matrices. Using the transformation $\Phi=\Psi\exp(-wt)$, one can see that Eq. (4) is equivalent to the one-dimensional Dirac equation in the Weyl representation

$$i\hbar\frac{\partial\Psi}{\partial t}=-ic\hbar\sigma_3\frac{\partial\Psi}{\partial x}+mc^2\sigma_1\Psi, \tag{5}$$

provided c is identified with the speed of light and w is analytically continued [2] to the imaginary value $-imc^2/\hbar$. The same phase transformation applied directly to Eq. (3) leads to the Klein-Gordon equation for each component of the vector Ψ.

The checkerboard model allows one to think about some aspects of quantum behavior in terms of an essentially classical stochastic model with an unphysical imaginary transition rate [4]. Furthermore, McKeon and Ord [5] have shown that an analytic continuation of the transition rate is not required to obtain the Dirac equation if the particle is allowed to move stochastically both forward and backward in time to mimic virtual pair creation and annihilation events. This approach has been further generalized [6] to incorporate an external field into Eq. (5).

The aim of this paper is to draw attention to a two-velocity-state model of another kind [7], which immediately leads to the potential-free Klein-Gordon equation without recourse to analytic continuation or backward-time motion. Let us extend the assumption that velocity reversals are Poisson distributed to the general case, when transitions between two velocity states are described by the arbitrary "field" $\xi(x,t)$,

$$\frac{\partial\phi_+(x,t)}{\partial t}=-c\frac{\partial\phi_+(x,t)}{\partial x}+\xi(x,t), \tag{6}$$

$$\frac{\partial\phi_-(x,t)}{\partial t}=c\frac{\partial\phi_-(x,t)}{\partial x}-\xi(x,t). \tag{7}$$

Addition and subtraction of Eqs. (6) and (7) yields

$$\frac{\partial\phi}{\partial t}=-c\frac{\partial\Delta(x,t)}{\partial x}, \tag{8}$$

$$\frac{\partial\Delta}{\partial t}=-c\frac{\partial\phi}{\partial x}+2\xi, \tag{9}$$

where we have defined

$$\phi(x,t)=\phi_+(x,t)+\phi_-(x,t),$$

$$\Delta(x,t)=\phi_+(x,t)-\phi_-(x,t).$$

Taking the time derivative of Eqs. (8) and (9) gives

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} - 2c\frac{\partial \xi}{\partial x}, \tag{10}$$

$$\frac{\partial^2 \Delta}{\partial t^2} = c^2 \frac{\partial^2 \Delta}{\partial x^2} + 2\frac{\partial \xi}{\partial t}. \tag{11}$$

If the field $\xi(x,t)$ satisfies the equation

$$\frac{\partial \xi(x,t)}{\partial x} = a\phi(x,t) \tag{12}$$

with the constant a chosen to be $a = m^2c^3/(2\hbar^2)$, then it follows from Eq. (10) that the function $\phi(x,t)$ satisfies the Klein-Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{m^2c^4}{\hbar^2}\phi. \tag{13}$$

Similarly, if the field $\xi(x,t)$ obeys the condition

$$\frac{1}{c}\frac{\partial \xi(x,t)}{\partial t} = -a\Delta(x,t) \tag{14}$$

with the same constant a as above, the Klein-Gordon equation for the function $\Delta(x,t)$ follows immediately from Eq. (11). When $\xi(x,t)$ satisfies both conditions (12) and (14) simultaneously, the Klein-Gordon equation holds for both functions $\phi(x,t)$ and $\Delta(x,t)$, which could be considered as components of a two-component vector field. Note that Eq. (8) connecting the components ϕ and Δ in the two-component field has the form of the Lorentz gauge.

It is difficult to interpret Eqs. (12) and (14) within the framework of the single-particle model. Clearly the function $\phi(x,t)$ cannot be interpreted as a probability density for a single-particle stochastic process since, in general, the solutions of the Klein-Gordon equation are not positive definite. On the other hand, the fact that $\phi(x,t)$ satisfies Eq. (8), which is of the form of a continuity equation, suggests the interpretation of the functions $\phi_{\pm}(x,t)$ as perturbations of charge densities in a many-body system. Below we outline one possible many-particle model in which Eqs. (12) and (14) have a clear physical meaning.

Consider a hypothetical plasmalike substance, which consists of an ensemble of identical particles of charge q and mass m moving in a uniform compensating background of the opposite charge. We assume that particles can move only along the x axis and their velocities can only take two values: c and $-c$. Suppose the dynamics of local densities $\Phi_+(x,t)$ and $\Phi_-(x,t)$ of particles traveling in the positive and negative directions, respectively, are determined by the equations

$$\frac{\partial \Phi_{\pm}(x,t)}{\partial t} = \mp c\frac{\partial \Phi_{\pm}(x,t)}{\partial x} + \gamma_{\pm}(x,t), \tag{15}$$

where the terms γ_+ and γ_- describe transitions at position x and time t between the two velocity states. In contrast to the original checkerboard model, where the velocity reversal is a Poisson-distributed stochastic process, we assume that the transitions are induced by the intrinsic field $E(x,t)$ resulting from local fluctuations of the net charge density around its equilibrium value of zero. If the particles interact through a Coloumb-like long-ranged force, the field $E(x,t)$ satisfies the Poisson equation

$$\frac{\partial E}{\partial x} = 4\pi q(\Phi - \Phi_0), \tag{16}$$

where Φ_0 is the constant density of the compensating background charge and $\Phi(x,t) = \Phi_- + \Phi_+$ is the total particle density.

Since the particles in the assembly interact through the force $F(x,t) = qE(x,t)$, but can only assume discrete velocities c and $-c$, a rule specifying how the force determines velocity reversal in the assembly must be postulated. It is possible to choose rules for the velocity reversal in such a way that densities of the system behave according to particular dynamical laws. Let us define the transition terms so that the momentum density $P(x,t) = mc[\Phi_+(x,t) - \Phi_-(x,t)]$ evolves according to classical Newtonian dynamics, i.e., its force-induced variation satisfies

$$\left(\frac{\partial P}{\partial t}\right)_f = qE\Phi. \tag{17}$$

The connection between the force and the transition terms $\gamma_{\pm}$ can be made by noting that the variation of the momentum density is a result of the velocity reversal, and hence

$$\left(\frac{\partial P}{\partial t}\right)_f = mc(\gamma_+ - \gamma_-). \tag{18}$$

Since $\gamma_+ = -\gamma_-$ for the two-velocity-state model, Eqs. (17) and (18) imply that the transition terms are determined by

$$\gamma_{\pm}(x,t) = \pm\frac{q\Phi(x,t)E(x,t)}{2mc}. \tag{19}$$

These rules can be implemented on the level of a single particle by stipulating that the velocity reversal occur with a probability determined by the transition term $\gamma_{\pm}$ when the force $F(x,t)$ and the particle velocity are in opposite directions and do not change when the force and velocity are in the same direction.

The model formulated above is dynamical in the sense that the transition rates between the two velocity states is completely specified by the intrinsic field $E(x,t)$. On the other hand, the motion of individual particles in the assembly is stochastic since only the probability of finding a particle in a particular velocity state is determined. Although the field $E(x,t)$ governs the transition terms at a given point, it does not depend on the particle index and hence any particle at the point has an equal probability to reverse its direction.

In equilibrium $\Phi_+(x,t) = \Phi_-(x,t) = \Phi_0/2$, the plasma has no net charge or current, and the field $E(x,t)$ is zero. Local fluctuations of the charge density from neutrality gives rise to excitations

$$\phi_{\pm}(x,t)=\Phi_{\pm}(x,t)-\Phi_0/2, \tag{20}$$

which can be represented as a superposition of waves with frequencies close to

$$\omega_p=\left(\frac{4\pi q^2\Phi_0}{m}\right)^{1/2}. \tag{21}$$

These waves correspond to electrostatic waves in a classical plasma, and, to linear order in the fluctuations, satisfy

$$\frac{\partial\phi_{\pm}}{\partial t}=\mp c\frac{\partial\phi_{\pm}}{\partial x}\pm\frac{q\Phi_0}{2mc}E, \tag{22}$$

which are just Eqs. (6) and (7) with

$$\xi(x,t)=\frac{q\Phi_0}{2mc}E(x,t). \tag{23}$$

With ξ given by Eq. (23), condition (12) is equivalent to the Poisson equation

$$\frac{\partial E}{\partial x}=4\pi q\phi, \tag{24}$$

provided that $a=2\pi q^2\phi_0/mc$. Furthermore, the second condition (14) takes the form

$$\frac{1}{4\pi}\frac{\partial E}{\partial t}=-qc\Delta(x,t), \tag{25}$$

which implies that the displacement current exactly cancels the drift current $j=qc\Delta$ associated with particle oscillation. This cancellation reflects the fact that electrostatic waves do not involve a magnetic field.

Equations (22), (24), and (25) lead to the Klein-Gordon equation for the charge fluctuation $\phi(x,t)$ and the current density $\Delta(x,t)$ provided the Debye screening length $\lambda_D=c/\omega_p$ of the plasma is taken to be the Compton wavelength λ_C of plasma particles,

$$\lambda_D=\left(\frac{mc^2}{4\pi q^2\Phi_0}\right)^{1/2}\leftrightarrow\lambda_C=\frac{\hbar}{mc}. \tag{26}$$

Thus, we see that there is a correspondence between coherent oscillations (plasma waves) of the assembly of interacting particles with two velocity states and the propagation of a free quantum particle in one dimension. The existence of such a correspondence is not surprising, since the approximate dispersion relation for electrostatic waves in a hot classical plasma has the same form as that for a relativistic quantum particle. The distinctive feature of a plasma composed of particles with fixed absolute velocities is that for this system the quadratic dispersion law $\omega^2(k)=\omega_p^2+c^2k^2$, and consequently the Klein-Gordon equation, exactly holds. Another important property of the two-state plasma model is that the velocity of particles is less than the phase velocity of waves $v_{ph}\approx c+(1/2c)(\omega_p/k)^2$, so Landau damping is absent and excitations do not decay.

Since the Debye length is the smallest length scale for the collective behavior in plasma, the above considerations are restricted to the case of long-wavelength excitations of the plasma with $\lambda>\lambda_D$. For shorter wavelengths, the coherent response of particles is suppressed and the plasma behaves like a system of individual particles. On the other hand, the Compton wavelength λ_C is the length scale on which the single-particle quantum equations becomes meaningless because of particle-antiparticle creation. In both cases, the length scales λ_D and λ_C give lower bounds for the validity of the equations, and their correspondence is remarkable and perhaps physically reasonable.

Relation (26) can be also written in the form of a condition for equilibrium density of the plasma

$$\Phi_0\leftrightarrow\frac{1}{4\pi}\alpha^{-1}\lambda_C^{-3}, \tag{27}$$

where $\alpha=q^2/\hbar c$ is the fine-structure constant. The density Φ_0 should be high enough so that the condition of collective behavior of the plasma $\Phi_0\lambda_D^3\gg1$ is satisfied. From relations (26) and (27), one can see that this condition is equivalent to $\alpha\ll1$.

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